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# Multi-dimensional BSDE with Oblique Reflection and Optimal Switching

Ying Hu<sup>\*</sup> and Shanjian Tang<sup>†</sup>

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## Abstract

In this paper, we study a multi-dimensional backward stochastic differential equation (BSDE) with oblique reflection, which is a BSDE reflected on the boundary of a special unbounded convex domain along an oblique direction, and which arises naturally in the study of optimal switching problem. The existence of the adapted solution is obtained by the penalization method, the monotone convergence, and the a priori estimations. The uniqueness is obtained by a verification method (the first component of any adapted solution is shown to be the vector value of a switching problem for BSDEs). As applications, we apply the above results to solve the optimal switching problem for stochastic differential equations of functional type, and we give also a probabilistic interpretation of the viscosity solution to a system of variational inequalities.

**Key Words.** Backward stochastic differential equations, oblique reflection, optimal switching, variational inequalities

**Abbreviated title.** Multi-dimensional BSDEs with oblique reflection

**AMS Subject Classifications.** 60H10, 93E20

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<sup>\*</sup>IRMAR, Université Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France. Part of this work was completed when this author was visiting Laboratory of Mathematics for Nonlinear Sciences, Fudan University, whose hospitality is greatly appreciated. *E-mail:* Ying.Hu@univ-rennes1.fr.

<sup>†</sup>Institute of Mathematics and Department of Finance and Control Sciences, School of Mathematical Sciences, Fudan University, Shanghai 200433, China. This author is supported in part by NSFC Grant #10325101, and the Chang Jiang Scholars Programme. Part of this work was completed when this author was visiting IRMAR, Université Rennes 1, whose hospitality is greatly appreciated. *E-mail:* sjtang@fudan.edu.cn.

# 1 Introduction

In this paper, we are concerned with the following reflected backward stochastic differential equation (RBSDE for short) with oblique reflection: for  $i \in \Lambda := \{1, \dots, m\}$ ,

$$\begin{cases} Y_i(t) = \xi_i + \int_t^T \psi(s, Y_i(s), Z_i(s), i) ds - \int_t^T dK_i(s) - \int_t^T Z_i(s) dW(s), & t \in [0, T], \\ Y_i(t) \leq \min_{j \neq i} \{Y_j(t) + k(i, j)\}, \\ \int_0^T \left( Y_i(s) - \min_{j \neq i} \{Y_j(s) + k(i, j)\} \right) dK_i(s) = 0. \end{cases} \quad (1.1)$$

Here,  $W$  is a standard Brownian motion on a complete probability space  $(\Omega, \mathcal{F}, P)$  and  $\xi$  is an  $m$ -dimensional random variable measurable with respect to the past of  $W$  up to time  $T$ .  $\xi$  is the terminal condition and  $\psi$  the coefficient (also called the generator).  $k$  is a real function defined on  $\Lambda \times \Lambda$ . The unknowns are the processes  $\{Y(t)\}_{t \in [0, T]}$ ,  $\{Z(t)\}_{t \in [0, T]}$ , and  $\{K(t)\}_{t \in [0, T]}$ , which are required to be adapted with respect to the natural completed filtration of the Brownian motion  $W$ . Moreover,  $K$  is an increasing process. The third relation in (1.1) is called the minimal boundary condition.

RBSDE (1.1) evolves in the closure  $\bar{Q}$  of domain  $Q$ :

$$Q := \{(y_1, \dots, y_m)^T \in \mathbb{R}^m : y_i < y_j + k(i, j) \text{ for any } i, j \in \Lambda \text{ such that } j \neq i\},$$

which is convex and unbounded. The boundary  $\partial Q$  of domain  $Q$  consists of the boundaries  $\partial L_i^-, i \in \Lambda$ , with

$$L_i^- := \{(y_1, \dots, y_m)^T \in \mathbb{R}^m : y_i < y_j + k(i, j), \text{ for any } j \in \Lambda \text{ such that } j \neq i\}, i \in \Lambda.$$

That is,

$$\partial Q = \bigcup_{i=1}^m \partial L_i^-.$$

In the interior of  $\bar{Q}$ , each equation in (1.1) is independent of others. On the boundary, say  $\partial L_i^-$ , the  $i$ -th equation is switched to another one, and the solution is reflected along the oblique direction  $-e_i$  (which is the opposite direction of the  $i$ -th coordinate axis).

RBSDE was first studied by El Karoui et al. [9] for the one-dimensional case. Multidimensional RBSDE was studied by Gegout-Petit and Pardoux [10], but their BSDE is reflected on the boundary of a convex domain along the inward normal direction, and their method depends heavily on the properties of this inward normal reflection (see (1)-(3) in [10]). We note that in a very special case (e.g.,  $\psi$  is independent of  $z$ ), Ramasubramanian [16] studied a BSDE in an orthant with oblique reflection. Note also that there are some papers dealing with SDEs with oblique reflection (see, e.g. [13] and [7]).

An incomplete and less general form of RBSDE (1.1) (where the minimal condition of (1.1) is missing and the generator  $\psi$  does not depend on  $(y, z)$ ) is suggested by [3].

But they did not discuss the existence and uniqueness of solution, which is considered to be difficult. See Remark 3.1 in [3].

Besides the theoretic interest, RBSDE (1.1) arises naturally from the following optimal switching problem.

Consider the switched equation

$$X^{a(\cdot)}(t) = x_0 + \int_0^t \sigma(s, X^{a(\cdot)})[dW(s) + b(s, X^{a(\cdot)}, a(s))ds], \quad t \in [0, T] \quad (1.2)$$

and the cost functional

$$J(a(\cdot)) = E\left[\int_0^T l(s, X^{a(\cdot)}, a(s))ds\right] + E\left[\sum_{i=1}^{\infty} k(\alpha_{i-1}, \alpha_i)\right]. \quad (1.3)$$

The optimal switching problem is to minimize the cost  $J(a(\cdot))$  with respect to  $a(\cdot)$ , subject to the state equation (1.2).

In the above,  $x_0$  is a fixed point in  $R^d$ .  $\sigma$ ,  $b$  and  $l$  are defined on  $[0, T] \times C([0, T]; R^d)$ ,  $[0, T] \times C([0, T]; R^d) \times \Lambda$  and  $[0, T] \times C([0, T]; R^d) \times \Lambda$ , respectively, with appropriate dimensions.

$$a(\cdot) = \alpha_0 \chi_{\{\theta_0\}}(\cdot) + \sum_{i=1}^{\infty} \alpha_{i-1} \chi_{(\theta_{i-1}, \theta_i]}(\cdot)$$

is called an admissible switching strategy if for any  $i$ ,  $\theta_i$  is a stopping time, and  $\alpha_i$  is an  $\mathcal{F}_{\theta_i}$ -measurable random variable with values in  $\Lambda$ . Here,  $\chi$  is the indicator function.  $k$  is called the switching cost.

Optimal switching is a special case of impulse control. The classical method of quasi-variational inequalities to solve impulse control problems driven by Markov processes is referred to the book of Bensoussan and Lions [2]. See [18] and the references therein for the theory of variational inequalities and the dynamic programming for optimal stochastic switching. But these works are restricted within the Markovian case. Recently, using the method of Snell envelope (see, e.g. El Karoui [8]) combined with the theory of scalar valued RBSDEs, Hamadene and Jeanblanc [11] studied the switching problem with two modes (i.e.,  $m = 2$ ) in the non-Markovian context. Djehiche, Hamadene and Popier [6] generalized their result to the above switching problem with multi modes. We note that in both [11] and [6], the drift term  $b$  does not depend on the switching strategy  $a$ .

The main contribution of this paper is to establish the existence and uniqueness of solution for RBSDE (1.1). We prove the existence by the penalization method, the monotone convergence, and the a priori estimation whose proof is rather technical. The proof of uniqueness is quite different: the classical method to estimate the difference of two solutions appears difficult to be applied to our present case of the oblique reflection. We obtain the uniqueness by a verification method: first we introduce an optimal switching problem for BSDEs, then we prove that the first component  $Y$  of any adapted solution  $(Y, Z, K)$  of RBSDE (1.1) is the (vector) value for the switching problem. As applications, we solve the optimal switching problem (1.2) and (1.3),

and we establish the Feynman-Kac formula for the viscosity solution to a system of variational inequalities.

The paper is organized as follows: in Section 2, we prove the existence of solution, whereas Section 3 is devoted to the uniqueness. We solve the optimal switching problem in Section 4. Finally, in Section 5, we give a probabilistic interpretation of the viscosity solution to a system of variational inequalities.

## 2 Existence

### 2.1 Notations

Let us fix a nonnegative real number  $T > 0$ . First of all,  $W = \{W_t\}_{t \geq 0}$  is a standard Brownian motion with values in  $R^d$  defined on some complete probability space  $(\Omega, \mathcal{F}, P)$ .  $\{\mathcal{F}_t, t \geq 0\}$  is the natural filtration of the Brownian motion  $W$  augmented by the  $P$ -null sets of  $\mathcal{F}$ . All the measurability notions will refer to this filtration. In particular, the sigma-field of predictable subsets of  $[0, T] \times \Omega$  is denoted by  $\mathcal{P}$ .

Let us consider now the RBSDE (1.1). The generator  $\psi$  is a random function  $\psi : [0, T] \times \Omega \times R \times R^d \times \Lambda \rightarrow R$  whose component  $\psi(\cdot, i)$  is measurable with respect to  $\mathcal{P} \otimes \mathcal{B}(R) \otimes \mathcal{B}(R^d)$  and the terminal condition  $\xi$  is simply a  $R^m$ -valued  $\mathcal{F}_T$ -measurable random variable.  $k$  is defined on  $\Lambda \times \Lambda$  and scalar valued.

By a solution to RBSDE (1.1) we mean a triple  $(Y, Z, K) = \{Y(t), Z(t), K(t)\}_{t \in [0, T]}$  of predictable processes with values in  $R^m \times R^{m \times d} \times R^m$  such that  $P$ -a.s.,  $t \rightarrow Y(t)$  and  $t \rightarrow K(t)$  are continuous,  $t \rightarrow Z(t)$  belongs to  $L^2(0, T)$ ,  $t \rightarrow \psi(t, Y_i(t), Z_i(t), i)$  belongs to  $L^1(0, T)$  and  $P$ -a.s., RBSDE (1.1) holds.

$S^2(R^m)$  or simply  $S^2$  denotes the set of  $R^m$ -valued, adapted and càdlàg processes  $\{Y(t)\}_{t \in [0, T]}$  such that

$$\|Y\|_{S^2} := E[\sup_{t \in [0, T]} |Y(t)|^2]^{1/2} < +\infty.$$

$(S^2, \|\cdot\|_{S^2})$  is a Banach space.

$M^2(R^{m \times d})$  or simply  $M^2$  denotes the set of (equivalent classes of) predictable processes  $\{Z(t)\}_{t \in [0, T]}$  with values in  $R^{m \times d}$  such that

$$\|Z\|_{M^2} := E[\int_0^T |Z(s)|^2 ds]^{1/2} < +\infty.$$

$M^2$  is a Banach space endowed with this norm.

$$\begin{aligned} N^2(R^m) : &= \{K = (K_1, \dots, K_m)^T \in S^2 : \text{for any } i \in \Lambda, K_i(0) = 0, \\ &\text{and } t \rightarrow K_i(t) \text{ is increasing} \}, \end{aligned}$$

where  $^T$  means transpose.  $(N^2, \|\cdot\|_{S^2})$  is a Banach space.

## 2.2 Existence

In this subsection, we prove the existence result for RBSDE (1.1). We assume the following Lipschitz condition on the generator.

**Hypothesis 2.1.** (i)  $\psi(\cdot, 0, 0) := (\psi(\cdot, 0, 0, 1), \dots, \psi(\cdot, 0, 0, m))^T$  belongs to  $M^2$ .

(ii) There exists a constant  $C > 0$ , such that,  $P$ -a.s. for each  $(t, y, y', z, z', i) \in [0, T] \times R \times R \times R^d \times R^d \times \Lambda$ ,

$$|\psi(t, y, z, i) - \psi(t, y', z', i)| \leq C(|y - y'| + |z - z'|).$$

We make the following assumption on  $k$  which is standard in the literature of optimal switching.

**Hypothesis 2.2.** (i) For any  $(i, j) \in \Lambda \times \Lambda$ ,  $k(i, j) \geq 0$ .

(ii) For any  $(i, j, l) \in \Lambda \times \Lambda \times \Lambda$ ,

$$k(i, j) + k(j, l) \geq k(i, l).$$

We are now in position to state the existence result.

**Theorem 2.1.** Let the Hypotheses 2.1 and 2.2 hold. Assume that  $\xi \in L^2(\Omega, \mathcal{F}_T, P; R^m)$  takes values in  $\bar{Q}$ . Then RBSDE (1.1) has a solution  $(Y, Z, K)$  in  $S^2 \times M^2 \times N^2$ .

We first sketch our proof.

**Sketch of the Proof:** The proof is divided to four steps. In Step 1, we introduce the penalized BSDEs whose existence and uniqueness follows from the classical result. In Step 2, we state some (uniform) a priori estimates for the solutions of penalized BSDEs, whose proof will be given in the next subsection. In Step 3, we prove the (monotone) convergence of these solutions. Finally, in Step 4, we check out the minimal boundary condition.

**Proof of Theorem 2.1 :** Step 1. The penalized BSDEs.

For any nonnegative integer  $n$ , let us introduce the following penalized BSDE:

$$\begin{aligned} Y_i^n(t) &= \xi_i + \int_t^T \psi(s, Y_i^n(s), Z_i^n(s), i) ds \\ &\quad - n \sum_{l=1}^m \int_t^T (Y_i^n(s) - Y_l^n(s) - k(i, l))^+ ds \\ &\quad - \int_t^T Z_i^n(s) dW(s), \quad t \in [0, T], \quad i \in \Lambda. \end{aligned} \tag{2.1}$$

Note that when  $l = i$ , we have, in view of Hypothesis 2.2 (i),

$$(Y_i^n(s) - Y_l^n(s) - k(i, l))^+ = 0. \tag{2.2}$$

From the classical result of Pardoux and Peng [14], for any  $n$ , BSDE (2.1) has a unique solution  $(Y^n, Z^n)$  in the space  $S^2 \times M^2$ .

Step 2. A priori estimates.

The following lemma will play a crucial rule in the proof of Theorem 2.1.

**Lemma 2.1.** *Let the Hypotheses 2.1 and 2.2 hold. Let us also assume that  $\xi \in L^2(\Omega, \mathcal{F}_T, P; R^m)$  takes values in  $\bar{Q}$ . Then there exists a constant  $C > 0$  (independent of  $n$ ), such that*

$$\|Y^n\|_{S^2} + \|Z^n\|_{M^2} \leq C \quad (2.3)$$

and

$$n^2 E \int_0^T ((Y_i^n(s) - Y_j^n(s) - k(i, j))^+)^2 ds \leq C. \quad (2.4)$$

However, the proof of this lemma is quite lengthy and delicate. We relegate it to the next subsection.

Step 3. Convergence of solutions  $\{Y^n, Z^n\}$  of penalized BSDEs.

First, for each  $n$ , we introduce a function  $\psi^n$  as follows:

$$\psi^n(t, y, z, i) := \psi(t, y_i, z_i, i) - n \sum_{l=1}^m (y_i - y_l - k(i, l))^+, \quad (t, y, z, i) \in [0, T] \times R^m \times R^{m \times d} \times \Lambda.$$

Since  $\psi^n(\cdot, z, i)$  depends only on  $z_i$  and

$$\psi^n(t, y + y', z_i, i) \geq \psi^n(t, y', z_i, i)$$

for any  $y \in R^m$  such that  $y \geq 0$  and the  $i$ -th component  $y_i = 0$ , it is easy to check

$$-4 \langle y^-, \psi^n(t, y^+ + y', z) - \psi^{n+1}(t, y', z') \rangle \leq 2 \sum_{i=1}^m \chi_{y_i < 0} |z_i - z'_i|^2 + C |y'|^2, \quad P - a.s.,$$

for a constant  $C > 0$ . We note that this is the inequality (5) in [12]. Applying the comparison theorem for multi-dimensional BSDEs (see Hu and Peng [12, Theorem 2.1]), we deduce that for any nonnegative integer  $n$ ,

$$Y_i^n(t) \geq Y_i^{n+1}(t), \forall i \in \Lambda, t \in [0, T]. \quad (2.5)$$

For a.e.  $t$  and  $P$ -a.s.  $\omega$ ,  $\{Y^n(t, \omega)\}_n$  admits a limit, denoted by  $Y(t, \omega)$ . Moreover from the a priori estimate (2.3) and Fatou's lemma, we have

$$\sup_{t \in [0, T]} E |Y(t)|^2 \leq C. \quad (2.6)$$

In view of the fact that  $Y_i(t) \leq Y_i^n(t) \leq Y_i^0(t)$  with  $i \in \Lambda$ , applying Lebesgue's dominated convergence theorem, we have

$$\lim_{n \rightarrow \infty} E \int_0^T |Y^n(s) - Y(s)|^2 ds = 0. \quad (2.7)$$

Now we prove that  $\{(Y^n, Z^n)\}_n$  is a Cauchy sequence in the space  $S^2 \times M^2$ . For this purpose, we apply Itô's formula to  $|Y_i^n(t) - Y_i^p(t)|^2$  to obtain

$$\begin{aligned}
& |Y_i^n(t) - Y_i^p(t)|^2 + \int_t^T |Z_i^n(s) - Z_i^p(s)|^2 ds \\
&= 2 \int_t^T (Y_i^n(s) - Y_i^p(s))(\psi(s, Y_i^n(s), Z_i^n(s), i) - \psi(s, Y_i^p(s), Z_i^p(s), i)) ds \\
&\quad - 2 \int_t^T (Y_i^n(s) - Y_i^p(s))n(Y_i^n(s) - Y_j^n(s) - k(i, j))^+ ds \\
&\quad + 2 \int_t^T (Y_i^n(s) - Y_i^p(s))p(Y_i^p(s) - Y_j^p(s) - k(i, j))^+ ds \\
&\quad - 2 \int_t^T (Y_i^n(s) - Y_i^p(s))(Z_i^n(s) - Z_i^p(s)) dW(s), i \in \Lambda. \tag{2.8}
\end{aligned}$$

Putting  $t = 0$  and taking expectation in the last equality, we get for  $i \in \Lambda$ ,

$$\begin{aligned}
& E|Y_i^n(0) - Y_i^p(0)|^2 + E \int_0^T |Z_i^n(s) - Z_i^p(s)|^2 ds \\
&= 2E \int_0^T (Y_i^n(s) - Y_i^p(s))(\psi(s, Y_i^n(s), Z_i^n(s), i) - \psi(s, Y_i^p(s), Z_i^p(s), i)) ds \\
&\quad - 2E \int_0^T (Y_i^n(s) - Y_i^p(s))n(Y_i^n(s) - Y_j^n(s) - k(i, j))^+ ds \\
&\quad + 2E \int_0^T (Y_i^n(s) - Y_i^p(s))p(Y_i^p(s) - Y_j^p(s) - k(i, j))^+ ds \\
&\leq CE \int_0^T (Y_i^n(s) - Y_i^p(s))^2 ds + \frac{1}{2}E \int_0^T |Z_i^n(s) - Z_i^p(s)|^2 ds \\
&\quad + \left( E \int_0^T |Y_i^n(s) - Y_i^p(s)|^2 ds \right)^{\frac{1}{2}} \left( E \int_0^T n^2 ((Y_i^n(s) - Y_j^n(s) - k(i, j))^+)^2 ds \right)^{\frac{1}{2}} \\
&\quad + \left( E \int_0^T |Y_i^n(s) - Y_i^p(s)|^2 ds \right)^{\frac{1}{2}} \left( E \int_0^T p^2 ((Y_i^p(s) - Y_j^p(s) - k(i, j))^+)^2 ds \right)^{\frac{1}{2}}.
\end{aligned}$$

From (2.4) and (2.7), we have

$$\lim_{n, p \rightarrow \infty} E \int_0^T |Z_i^n(s) - Z_i^p(s)|^2 ds = 0, i \in \Lambda. \tag{2.9}$$

Now, we define the increasing process  $K_i^n$  as follows:

$$K_i^n(t) := n \int_0^t \sum_{l=1}^m (Y_i^n(s) - Y_l^n(s) - k(i, l))^+ ds, \quad t \in [0, T], i \in \Lambda. \tag{2.10}$$

From the penalized BSDE (2.1), we have

$$K_i^n(t) = Y_i^n(t) - Y_i^n(0) + \int_0^t \psi(s, Y_i^n(s), Z_i^n(s), i) ds - \int_0^t Z_i^n(s) dW(s), i \in \Lambda. \tag{2.11}$$

We denote by  $Z$  the limit of  $Z^n$  in  $M^2$ . Set

$$K_i(t) := Y_i(t) - Y_i(0) + \int_0^t \psi(s, Y_i(s), Z_i(s), i) ds - \int_0^t Z_i(s) dW(s), i \in \Lambda. \tag{2.12}$$



We have

$$\lim_{n \rightarrow \infty} \|Z^n - Z\|_{M^2} = 0.$$

Going back again to (2.8), we deduce that for  $i \in \Lambda$ ,

$$\begin{aligned} & E \sup_{0 \leq t \leq T} |Y_i^n(t) - Y_i^p(t)|^2 \\ \leq & 2E \int_0^T |Y_i^n(s) - Y_i^p(s)| |\psi(s, Y_i^n(s), Z_i^n(s), i) - \psi(s, Y_i^p(s), Z_i^p(s), i)| ds \\ & + 2E \int_0^T |Y_i^n(s) - Y_i^p(s)| n(Y_i^n(s) - Y_j^n(s) - k(i, j))^+ ds \\ & + 2E \int_0^T |Y_i^n(s) - Y_i^p(s)| p(Y_i^p(s) - Y_j^p(s) - k(i, j))^+ ds \\ & + 2E \sup_{0 \leq t \leq T} \left| \int_t^T (Y_i^n(s) - Y_i^p(s))(Z_i^n(s) - Z_i^p(s)) dW(s) \right|. \end{aligned} \quad (2.13)$$

The last term of the right hand side of the last inequality is less than or equal to the following quantity:

$$\begin{aligned} & CE \left( \int_0^T |(Y_i^n(s) - Y_i^p(s))(Z_i^n(s) - Z_i^p(s))|^2 ds \right)^{\frac{1}{2}} \\ \leq & CE \left( \sup_{0 \leq t \leq T} |Y_i^n(t) - Y_i^p(t)| \int_0^T |Z_i^n(s) - Z_i^p(s)|^2 ds \right)^{\frac{1}{2}} \\ \leq & \frac{1}{2} E \sup_{0 \leq t \leq T} |Y_i^n(t) - Y_i^p(t)|^2 + CE \int_0^T |Z_i^n(s) - Z_i^p(s)|^2 ds. \end{aligned} \quad (2.14)$$

Combining (2.13) and (2.14) and taking into consideration (2.7) and (2.9), we deduce that  $\{Y^n\}_n$  is a Cauchy sequence in  $S^2$ , which means in particular that

$$\lim_{n \rightarrow \infty} \|Y^n - Y\|_{S^2} = 0.$$

Consequently,  $Y$  is a continuous process.

From (2.11), (2.12), and the following fact that

$$\|Y^n - Y\|_{S^2} + \|Z^n - Z\|_{M^2} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

we deduce immediately that

$$\lim_{n \rightarrow \infty} \|K^n - K\|_{S^2} = 0.$$

Hence,  $K \in N^2$ , and  $(Y, Z, K)$  satisfies the first relation in RBSDE (1.1).

Finally, from the a priori estimate (2.4), we have

$$E \int_0^T ((Y_i^n(s) - Y_j^n(s) - k(i, j))^+)^2 ds \leq \frac{C}{n^2}, \quad i, j \in \Lambda.$$

Letting  $n \rightarrow \infty$ , we deduce

$$E \int_0^T ((Y_i(s) - Y_j(s) - k(i, j))^+)^2 ds = 0, \quad i, j \in \Lambda.$$

Hence,

$$Y_i(s) \leq Y_j(s) + k(i, j), \quad s \in [0, T], i, j \in \Lambda, \quad (2.15)$$

which are equivalent to the following:

$$P - a.s. \quad Y(s) \in \bar{Q}, \quad \forall s \in [0, T].$$

Step 4. The minimal boundary condition.

Let us first state the following lemma whose proof is at the end of this subsection.

**Lemma 2.2.** *Let the Hypotheses 2.1 and 2.2 hold. Let us also assume that  $\xi \in L^2(\Omega, \mathcal{F}_T, P; R^m)$  takes values in  $\bar{Q}$ . We have, for any integer  $n$ ,*

$$\int_0^T \left( Y_i^n(s) - \min_{j \neq i} [Y_j^n(s) + k(i, j)] \right)^- dK_i^n(s) = 0, \quad i \in \Lambda. \quad (2.16)$$

Now, we can take the limit in (2.16) by letting  $n$  tend to  $+\infty$  and applying Lemma 5.8 in [10] to get the following

$$\int_0^T \left( Y_i(s) - \min_{j \neq i} [Y_j(s) + k(i, j)] \right)^- dK_i(s) = 0, \quad i \in \Lambda, \quad (2.17)$$

which, together with (2.15), yields the minimal boundary conditions.

The proof of Theorem 2.1 is now complete.  $\square$

**Proof of Lemma 2.2** For  $i \in \Lambda$ , the left hand side of the last equality is equal to the following sum

$$n \sum_{l=1}^m \int_0^T \min_{j \neq i} \left\{ (Y_i^n(s) - Y_j^n(s) - k(i, j))^- (Y_i^n(s) - Y_l^n(s) - k(i, l))^+ \right\} ds. \quad (2.18)$$

We claim that the integrand of the  $l$ -th integral is equal to zero for  $l \in \Lambda$ .

In fact, it is immediate for the case of  $l = i$ . For the case of  $l \neq i$ , the integrand is the minimum of the following  $m - 1$  nonnegative quantities:

$$(Y_i^n(s) - Y_j^n(s) - k(i, j))^- (Y_i^n(s) - Y_l^n(s) - k(i, l))^+, \quad j \in \Lambda \text{ and } j \neq i, \quad (2.19)$$

whose  $l$ -th term is zero due to the fact that

$$(Y_i^n(s) - Y_l^n(s) - k(i, l))^- (Y_i^n(s) - Y_l^n(s) - k(i, l))^+ = 0, \quad (2.20)$$

and therefore, it is zero. The desired result then follows.  $\square$

## 2.3 Proof of Lemma 2.1

In this subsection, we prove Lemma 2.1.

For  $i, j \in \Lambda$ , applying Tanaka's formula (see, e.g. [17]) to  $(Y_i^n(t) - Y_j^n(t) - k(i, j))^+$ , we have

$$\begin{aligned}
& (Y_i^n(t) - Y_j^n(t) - k(i, j))^+ + n \sum_{l=1}^m \int_t^T \chi_{\mathcal{L}_{ij,n}^+}(s) (Y_i^n(s) - Y_l^n(s) - k(i, l))^+ ds \\
& - n \sum_{l=1}^m \int_t^T \chi_{\mathcal{L}_{ij,n}^+}(s) (Y_j^n(s) - Y_l^n(s) - k(j, l))^+ ds + \frac{1}{2} \int_t^T dL_{ij}^n(s) \\
& = \int_t^T \chi_{\mathcal{L}_{ij,n}^+}(s) (\psi(s, Y_i^n(s), Z_i^n(s), i) - \psi(s, Y_j^n(s), Z_j^n(s), j)) ds \\
& - \int_t^T \chi_{\mathcal{L}_{ij,n}^+}(s) (Z_i^n(s) - Z_j^n(s)) dW(s),
\end{aligned} \tag{2.21}$$

where for  $i, j \in \Lambda$ ,

$$\mathcal{L}_{ij,n}^+ := \{(s, \omega) : Y_i^n(s) > Y_j^n(s) + k(i, j)\}, \tag{2.22}$$

and  $L_{ij}^n$  is the local time of the process  $Y_i^n - Y_j^n - k(i, j)$  at 0.

Applying Itô's formula to  $\left((Y_i^n(t) - Y_j^n(t) - k(i, j))^+\right)^2$  and taking into consideration

$$\int_t^T (Y_i^n(s) - Y_j^n(s) - k(i, j))^+ dL_{ij}^n(s) = 0, \forall t \in [0, T], \tag{2.23}$$

we have

$$\begin{aligned}
& \left((Y_i^n(t) - Y_j^n(t) - k(i, j))^+\right)^2 \\
& + 2n \int_t^T \left((Y_i^n(s) - Y_j^n(s) - k(i, j))^+\right)^2 ds \\
& + \int_t^T \chi_{\mathcal{L}_{ij,n}^+}(s) |Z_i^n(s) - Z_j^n(s)|^2 ds \\
& = 2 \int_t^T (Y_i^n(s) - Y_j^n(s) - k(i, j))^+ [\psi(s, Y_i^n(s), Z_i^n(s), i) - \psi(s, Y_j^n(s), Z_j^n(s), j)] ds \\
& - 2 \int_t^T (Y_i^n(s) - Y_j^n(s) - k(i, j))^+ (Z_i^n(s) - Z_j^n(s)) dW(s) \\
& + 2n \int_t^T (Y_i^n(s) - Y_j^n(s) - k(i, j))^+ (Y_j^n(s) - Y_i^n(s) - k(j, i))^+ ds \\
& + 2n \sum_{l \neq i, l \neq j} \int_t^T (Y_i^n(s) - Y_j^n(s) - k(i, j))^+ \\
& \times [(Y_j^n(s) - Y_l^n(s) - k(j, l))^+ - (Y_i^n(s) - Y_l^n(s) - k(i, l))^+] ds.
\end{aligned} \tag{2.24}$$

We claim that the integrands of the integrals in the last two terms of (2.24) are all less than or equal to zero. In fact, since

$$\{(y_1, \dots, y_m)^T \in R^m : y_i - y_j - k(i, j) > 0, y_j - y_i - k(j, i) > 0\} = \emptyset$$

due to the fact that

$$k(i, j) + k(j, i) \geq 0,$$

we have

$$(Y_i^n(s) - Y_j^n(s) - k(i, j))^+ (Y_j^n(s) - Y_i^n(s) - k(j, i))^+ = 0, \quad i, j \in \Lambda. \quad (2.25)$$

On the other hand, for  $l, i, j \in \Lambda$ , taking into consideration both Hypothesis 2.2 (ii), i.e.,

$$k(i, j) + k(j, l) \geq k(i, l),$$

and the elementary inequality that  $x_1^+ - x_2^+ \leq (x_1 - x_2)^+$  for any two real numbers  $x_1$  and  $x_2$ , we have

$$\begin{aligned} & (Y_i^n(s) - Y_j^n(s) - k(i, j))^+ [(Y_j^n(s) - Y_l^n(s) - k(j, l))^+ - (Y_i^n(s) - Y_l^n(s) - k(i, l))^+] \\ & \leq (Y_i^n(s) - Y_j^n(s) - k(i, j))^+ (Y_j^n(s) - Y_i^n(s) - k(j, l) + k(i, l))^+ \\ & \leq (Y_i^n(s) - Y_j^n(s) - k(i, l) + k(j, l))^+ (Y_j^n(s) - Y_i^n(s) - k(j, l) + k(i, l))^+. \end{aligned} \quad (2.26)$$

The last term of the last inequality is zero, since

$$\{(y_1, \dots, y_m)^T \in R^m : y_i - y_j - k(i, l) + k(j, l) > 0, y_j - y_i - k(j, l) + k(i, l) > 0\} = \emptyset.$$

Concluding the above, we have

$$\begin{aligned} & E \left( (Y_i^n(t) - Y_j^n(t) - k(i, j))^+ \right)^2 \\ & + 2nE \int_t^T ((Y_i^n(s) - Y_j^n(s) - k(i, j))^+)^2 ds \\ & + E \int_t^T \chi_{\mathcal{L}_{ij,n}^+}(s) |Z_i^n(s) - Z_j^n(s)|^2 ds \\ & \leq 2E \int_t^T (Y_i^n(s) - Y_j^n(s) - k(i, j))^+ |\psi(s, Y_i^n(s), Z_i^n(s), i) - \psi(s, Y_j^n(s), Z_j^n(s), j)| ds. \end{aligned} \quad (2.27)$$

In view of Hypothesis 2.1 on the function  $\psi$ , we have

$$\begin{aligned} & |\psi(s, Y_i^n(s), Z_i^n(s), i) - \psi(s, Y_j^n(s), Z_j^n(s), j)| \\ & \leq |\psi(s, Y_i^n(s), Z_i^n(s), i) - \psi(s, Y_i^n(s), Z_i^n(s), j)| \\ & \quad + |\psi(s, Y_i^n(s), Z_i^n(s), j) - \psi(s, Y_j^n(s), Z_j^n(s), j)| \\ & \leq C(|\psi(s, 0, 0)| + |Y_i^n(s)| + |Z_i^n(s)| + |Y_i^n(s) - Y_j^n(s)| + |Z_i^n(s) - Z_j^n(s)|) \\ & \leq C(1 + |\psi(s, 0, 0)| + |Y_i^n(s)| + |Z_i^n(s)| \\ & \quad + |Y_i^n(s) - Y_j^n(s) - k(i, j)| + |Z_i^n(s) - Z_j^n(s)|). \end{aligned} \quad (2.28)$$

Consequently, we have

$$\begin{aligned}
& E \left( (Y_i^n(t) - Y_j^n(t) - k(i, j))^+ \right)^2 \\
& + 2nE \int_t^T \left( (Y_i^n(s) - Y_j^n(s) - k(i, j))^+ \right)^2 ds \\
& + E \int_t^T \chi_{\mathcal{L}_{ij,n}^+}(s) |Z_i^n(s) - Z_j^n(s)|^2 ds \\
& \leq CE \int_t^T |(Y_i^n(s) - Y_j^n(s) - k(i, j))^+|^2 ds \\
& + \frac{1}{2}E \int_t^T \chi_{\mathcal{L}_{ij,n}^+}(s) (1 + |\psi(s, 0, 0)|^2 + |Y_i^n(s)|^2 + |Z_i^n(s)|^2 \\
& + |(Y_i^n(s) - Y_j^n(s) - k(i, j))^+|^2 + |Z_i^n(s) - Z_j^n(s)|^2) ds.
\end{aligned} \tag{2.29}$$

Applying Gronwall's inequality, we deduce easily that

$$\begin{aligned}
E \left( (Y_i^n(t) - Y_j^n(t) - k(i, j))^+ \right)^2 & \leq C \left( 1 + E \int_0^T \chi_{\mathcal{L}_{ij,n}^+}(s) (|Y_i^n(s)|^2 + |Z_i^n(s)|^2) ds \right), \\
nE \int_0^T \left( (Y_i^n(s) - Y_j^n(s) - k(i, j))^+ \right)^2 ds & + E \int_0^T \chi_{\mathcal{L}_{ij,n}^+}(s) |Z_i^n(s) - Z_j^n(s)|^2 ds \\
& \leq C \left( 1 + E \int_0^T \chi_{\mathcal{L}_{ij,n}^+}(s) [|Y_i^n(s)|^2 + |Z_i^n(s)|^2] ds \right).
\end{aligned} \tag{2.30}$$

Going back to (2.24) and applying Burkholder-Davis-Gundy's inequality, we obtain

$$E \left[ \sup_{0 \leq t \leq T} \left( (Y_i^n(t) - Y_j^n(t) - k(i, j))^+ \right)^2 \right] \leq C \left( 1 + E \int_0^T \chi_{\mathcal{L}_{ij,n}^+}(s) [|Y_i^n(s)|^2 + |Z_i^n(s)|^2] ds \right). \tag{2.31}$$

On the other hand, from (2.27), we deduce that,

$$\begin{aligned}
& E \left( (Y_i^n(t) - Y_j^n(t) - k(i, j))^+ \right)^2 + 2nE \int_t^T \left( (Y_i^n(s) - Y_j^n(s) - k(i, j))^+ \right)^2 ds \\
& + E \int_t^T \chi_{\mathcal{L}_{ij,n}^+}(s) |Z_i^n(s) - Z_j^n(s)|^2 ds \\
& \leq (n + C)E \int_t^T \left( (Y_i^n(s) - Y_j^n(s) - k(i, j))^+ \right)^2 ds \\
& + \frac{C}{n}E \int_t^T \chi_{\mathcal{L}_{ij,n}^+}(s) (1 + |\psi(s, 0, 0)|^2 + |Y_i^n(s)|^2 + |Z_i^n(s)|^2 + |Z_i^n(s) - Z_j^n(s)|^2) ds.
\end{aligned}$$

This shows that, for sufficiently large  $n$ ,

$$n^2 E \int_0^T \left( (Y_i^n(s) - Y_j^n(s) - k(i, j))^+ \right)^2 ds \leq C \left( 1 + E \int_0^T [|Y_i^n(s)|^2 + |Z_i^n(s)|^2] ds \right). \tag{2.32}$$

Finally, applying Itô's formula to  $|Y_i^n(t)|^2$ , we obtain:

$$\begin{aligned}
& |Y_i^n(t)|^2 + \int_t^T |Z_i^n(s)|^2 ds \\
= & |\xi_i|^2 + 2 \int_t^T Y_i^n(s) \cdot \left[ \psi(s, Y_i^n(s), Z_i^n(s), i) - \sum_{l=1}^m n(Y_i^n(s) - Y_l^n(s) - k(i, l))^+ \right] ds \\
& - 2 \int_t^T Z_i^n(s) dW(s). \tag{2.33}
\end{aligned}$$

By taking expectation and using the elementary inequality:

$$2ab \leq \frac{1}{\epsilon} a^2 + \epsilon b^2, \quad \forall \epsilon > 0,$$

we deduce that, for any  $\epsilon > 0$ ,

$$\begin{aligned}
& E|Y_i^n(t)|^2 + E \int_t^T |Z_i^n(s)|^2 ds \\
& \leq E|\xi_i|^2 + 2E \int_t^T |Y_i^n(s)| \cdot [|\psi(s, Y_i^n(s), Z_i^n(s), i)| \\
& \quad + \sum_{l=1}^m n(Y_i^n(s) - Y_l^n(s) - k(i, l))^+] ds \\
& \leq C + 2E \int_t^T |Y_i^n(s)| \cdot [C(|\psi(s, 0, 0)| + |Y_i^n(s)| + |Z_i^n(s)|) \\
& \quad + \sum_{l=1}^m n(Y_i^n(s) - Y_l^n(s) - k(i, l))^+] ds \\
& \leq C + C_\epsilon E \int_t^T |Y_i^n(s)|^2 ds + \epsilon E \int_t^T |Z_i^n(s)|^2 ds \\
& \quad + \epsilon E \int_t^T n^2 \sum_{l=1}^m ((Y_i^n(s) - Y_l^n(s) - k(i, l))^+)^2 ds \\
& \leq C_\epsilon + C_\epsilon E \int_t^T |Y_i^n(s)|^2 ds + C\epsilon E \int_t^T |Z_i^n(s)|^2 ds.
\end{aligned}$$

Here,  $C_\epsilon > 0$  denotes a constant which depends on  $\epsilon$  and may vary from line to line.

Therefore,

$$E|Y_i^n(t)|^2 + E \int_t^T |Z_i^n(s)|^2 ds \leq C. \tag{2.34}$$

From (2.32), we obtain (2.4), and from (2.33), we deduce

$$\|Y^n\|_{S^2} \leq C. \tag{2.35}$$

The proof of Lemma 2.1 is now complete.  $\square$

### 3 Uniqueness

In this section, we prove the uniqueness by a verification method. Let  $(\tilde{Y}, \tilde{Z}, \tilde{K})$  be a solution in the space  $(S^2, M^2, N^2)$  to RBSDE (1.1). We will prove that  $\tilde{Y}$  is in fact the (vector) value for an optimal switching problem of BSDEs. For this purpose, we introduce the following optimal switching problem.

Let  $\{\theta_j\}_{j=0}^\infty$  be an increasing sequence of stopping times with values in  $[0, T]$  and  $\forall j$ ,  $\alpha_j$  is an  $\mathcal{F}_{\theta_j}$ -measurable random variable with values in  $\Lambda$ , and  $\chi$  is the indicator function. We assume moreover that for  $P$ -a.s.  $\omega$ , there exists an integer  $N(\omega)$  such that  $\theta_N = T$ .

Then we define the admissible switching strategy as follows:

$$a(s) = \alpha_0 \chi_{\{\theta_0\}}(s) + \sum_{j=1}^N \alpha_{j-1} \chi_{(\theta_{j-1}, \theta_j]}(s). \quad (3.1)$$

We denote by  $\mathcal{A}$  the set of all these admissible switching strategies and by  $\mathcal{A}^i$  the subset of  $\mathcal{A}$  consisting of admissible switching strategies starting from the mode  $i$ . In the same way, we denote by  $\mathcal{A}_t$  the set of all the admissible strategies starting at the time  $t$  (or equivalently  $\theta_0 = t$ ) and by  $\mathcal{A}_t^i$  the subset of  $\mathcal{A}_t$  consisting of admissible switching strategies starting at time  $t$  from the mode  $i$ .

For any  $a(\cdot) \in \mathcal{A}_t$ , we define the associated (cost) process  $A^{a(\cdot)}$  as follows:

$$A^{a(\cdot)}(s) = \sum_{j=1}^{N-1} k(\alpha_{j-1}, \alpha_j) \chi_{[\theta_j, T]}(s), \quad s \in [t, T]. \quad (3.2)$$

Obviously,  $A^{a(\cdot)}(\cdot)$  is a càdlàg process.

Now we are in position to introduce the switched BSDE:

$$U(s) = \xi_{a(T)} + A^{a(\cdot)}(T) - A^{a(\cdot)}(s) + \int_s^T \psi(r, U(r), V(r), a(r)) dr - \int_s^T V(r) dW(r), \quad s \in [t, T]. \quad (3.3)$$

This is a (slightly) generalized BSDE: it is equivalent to the following standard BSDE:

$$\bar{U}(s) = \xi_{a(T)} + A^{a(\cdot)}(T) + \int_s^T \psi(r, \bar{U}(r) - A^{a(\cdot)}(r), \bar{V}(r), a(r)) dr - \int_s^T \bar{V}(r) dW(r), \quad s \in [t, T] \quad (3.4)$$

via the simple change of variable:

$$\bar{U}(s) = U(s) + A^{a(\cdot)}(s), \quad \bar{V}(s) = V(s), \quad s \in [t, T].$$

Hence, BSDE (3.3) has a solution in  $S^2 \times M^2$ . We denote this solution by  $(U^{a(\cdot)}, V^{a(\cdot)})$ . Note that  $U$  is only a càdlàg process.

The optimal switching problem with the initial mode  $i \in \Lambda$  is to minimize  $U^{a(\cdot)}(t)$  subject to  $a(\cdot) \in \mathcal{A}_t^i$ .

The assumptions required for the uniqueness will be slightly stronger than those needed for existence. We keep the same assumption on  $\psi$  and we assume the following for  $k$ .

**Hypothesis 3.1.** (i) For any  $(i, j) \in \Lambda \times \Lambda$ ,  $k(i, j) \geq 0$ .

(ii) For any  $(i, j, l) \in \Lambda \times \Lambda \times \Lambda$  such that  $i \neq j$  and  $j \neq l$ ,

$$k(i, j) + k(j, l) > k(i, l).$$

We have the following representation for the first component of the adapted solution to RBSDE (1.1), which immediately implies the uniqueness of the adapted solution to RBSDE (1.1).

**Theorem 3.1.** Let us suppose that the Hypotheses 2.1 and 3.1 hold. Let us also assume that  $\xi \in L^2(\Omega, \mathcal{F}_T, P; R^m)$  takes values in  $\bar{Q}$ . Let  $(\tilde{Y}, \tilde{Z}, \tilde{K})$  be a solution in  $(S^2, M^2, K^2)$  to RBSDE (1.1). Then

(i) For any  $a(\cdot) \in \mathcal{A}_t^i$ , we have:

$$\tilde{Y}_i(t) \leq U^{a(\cdot)}(t), \quad P - a.s. \quad (3.5)$$

(ii) Set  $\theta_0^* = t$ ,  $\alpha_0^* = i$ . We define the sequence  $\{\theta_j^*, \alpha_j^*\}_{j=1}^\infty$  in an inductive way as follows:

$$\theta_j^* := \inf \{s \geq \theta_{j-1}^* : \tilde{Y}_{\alpha_{j-1}^*}(s) = \min_{l \neq \alpha_{j-1}^*} \{\tilde{Y}_l(s) + k(\alpha_{j-1}^*, l)\} \wedge T, \quad (3.6)$$

and  $\alpha_j^*$  is the  $\mathcal{F}_{\theta_j^*}$ -measurable random variable such that

$$\tilde{Y}_{\alpha_{j-1}^*}(\theta_j^*) = \tilde{Y}_{\alpha_j^*}(\theta_j^*) + k(\alpha_{j-1}^*, \alpha_j^*),$$

with  $j = 1, 2, \dots$ .

Then,  $P$ -a.s.  $\omega$ , there exists an integer  $N(\omega)$  such that  $\theta_N^* = T$ . And the following switching strategy:

$$a^*(s) = i\chi_{\{t\}}(s) + \sum_{j=1}^N \alpha_{j-1}^* \chi_{(\theta_{j-1}^*, \theta_j^*]}(s), \quad (3.7)$$

is admissible, i.e.,  $a^*(\cdot) \in \mathcal{A}_t^i$ . Moreover,

$$\tilde{Y}_i(t) = U^{a^*(\cdot)}(t).$$

(iii) We have the following representation for  $\tilde{Y}(t)$ :

$$\tilde{Y}_i(t) = \operatorname{essinf}_{a(\cdot) \in \mathcal{A}_t^i} U^{a(\cdot)}(t), \quad i \in \Lambda, t \in [0, T].$$

RBSDE (1.1) has a unique solution.

**Proof.** Without loss of generality, we will prove (i) and (ii) for the case of  $t = 0$ . Otherwise, it suffices to consider the admissible switching strategies starting at time  $t$ .



(i) We define

$$\tilde{Y}^{a(\cdot)}(s) = \sum_{i=1}^N \tilde{Y}_{\alpha_{i-1}}(s) \chi_{[\theta_{i-1}, \theta_i)}(s) + \xi_{a(T)} \chi_{\{T\}}(s), \quad (3.8)$$

$$\tilde{Z}^{a(\cdot)}(s) = \sum_{i=1}^N \tilde{Z}_{\alpha_{i-1}}(s) \chi_{[\theta_{i-1}, \theta_i)}(s), \quad (3.9)$$

$$\tilde{K}^{a(\cdot)}(s) = \sum_{i=1}^N \int_{\theta_{i-1} \wedge s}^{\theta_i \wedge s} d\tilde{K}_{\alpha_{i-1}}(r). \quad (3.10)$$

Noting that  $\tilde{Y}^{a(\cdot)}(\cdot)$  is a càdlàg process with jump  $\tilde{Y}_{\alpha_i}(\theta_i) - \tilde{Y}_{\alpha_{i-1}}(\theta_i)$  at  $\theta_i$ ,  $i = 1, \dots, N-1$ , we deduce that

$$\begin{aligned} & \tilde{Y}^{a(\cdot)}(s) - \tilde{Y}^{a(\cdot)}(0) \\ &= \sum_{i=1}^N \int_{\theta_{i-1} \wedge s}^{\theta_i \wedge s} [-\psi(r, \tilde{Y}_{\alpha_{i-1}}(r), \tilde{Z}_{\alpha_{i-1}}(r), \alpha_{i-1}) dr + \tilde{Z}_{\alpha_{i-1}}(r) dW(r) + d\tilde{K}_{\alpha_{i-1}}(r)] \\ & \quad + \sum_{i=1}^{N-1} [\tilde{Y}_{\alpha_i}(\theta_i) - \tilde{Y}_{\alpha_{i-1}}(\theta_i)] \chi_{[\theta_i, T]}(s) \\ &= \int_0^s [-\psi(r, \tilde{Y}^{a(\cdot)}(r), \tilde{Z}^{a(\cdot)}(r), a(r)) dr + \tilde{Z}^{a(\cdot)}(r) dW(r) + d\tilde{K}^{a(\cdot)}(r)] \\ & \quad + \tilde{A}^{a(\cdot)}(s) - A^{a(\cdot)}(s), \end{aligned}$$

where

$$\tilde{A}^{a(\cdot)}(s) = \sum_{i=1}^{N-1} [\tilde{Y}_{\alpha_i}(\theta_i) + k(\alpha_{i-1}, \alpha_i) - \tilde{Y}_{\alpha_{i-1}}(\theta_i)] \chi_{[\theta_i, T]}(s), \quad (3.11)$$

and it is an increasing process due to the fact that

$$\tilde{Y}(t) \in \bar{Q}, \quad \forall t \in [0, T].$$

Consequently, we conclude that  $(\tilde{Y}^{a(\cdot)}, \tilde{Z}^{a(\cdot)})$  is a solution of the following BSDE:

$$\begin{aligned} & \tilde{Y}^{a(\cdot)}(s) \\ &= \xi_{a(T)} + A^{a(\cdot)}(T) - A^{a(\cdot)}(s) - [(\tilde{K}^{a(\cdot)}(T) + \tilde{A}^{a(\cdot)}(T)) - (\tilde{K}^{a(\cdot)}(s) + \tilde{A}^{a(\cdot)}(s))] \\ & \quad + \int_s^T \psi(r, \tilde{Y}^{a(\cdot)}(r), \tilde{Z}^{a(\cdot)}(r), a(r)) dr - \int_s^T \tilde{Z}^{a(\cdot)}(r) dW(r), \quad s \in [0, T]. \end{aligned} \quad (3.12)$$

Since both  $\tilde{K}^{a(\cdot)}$  and  $\tilde{A}^{a(\cdot)}$  are increasing càdlàg processes, from the comparison theorem, we conclude that

$$\tilde{Y}^{a(\cdot)}(0) \leq U^{a(\cdot)}(0),$$

which implies that

$$\tilde{Y}_i(0) \leq U^{a(\cdot)}(0).$$

(ii) Let us first claim that if  $0 \leq \theta_1^* < \theta_2^* < T$ , then there exists a constant  $c > 0$  such that

$$|\tilde{Y}(\theta_2^*) - \tilde{Y}(\theta_1^*)| \geq c.$$

To prove this claim, we introduce the following subsets of  $\bar{Q}$ : for  $i \neq j$ ,

$$B_{i,j} := \{(y_1, \dots, y_m)^T \in R^m : y_i = y_j + k(i, j)\} \cap \bar{Q}.$$

We assert that for  $i \neq j$  and  $j \neq l$ ,  $B_{i,j} \cap B_{j,l} = \emptyset$ .

In fact, if there exists an element  $(y_1, \dots, y_m)^T \in B_{i,j} \cap B_{j,l}$ , then

$$y_i = y_j + k(i, j) \text{ and } y_j = y_l + k(j, l).$$

We deduce then

$$y_i = y_l + k(i, j) + k(j, l) > y_l + k(i, l),$$

which contradicts the fact that  $(y_1, \dots, y_m)^T \in \bar{Q}$ .

Hence, the distance between  $B_{i,j}$  and  $B_{j,l}$  is strictly positive,

$$\text{dist}(B_{i,j}, B_{j,l}) > 0.$$

We set

$$c := \min_{i \neq j, j \neq l} \text{dist}(B_{i,j}, B_{j,l}) > 0.$$

We return to the proof of the claim. From the definition of  $(\theta_1^*, \alpha_1^*)$  and  $(\theta_2^*, \alpha_2^*)$ ,

$$\tilde{Y}(\theta_1^*) \in B_{i, \alpha_1^*} \text{ and } \tilde{Y}(\theta_2^*) \in B_{\alpha_1^*, \alpha_2^*},$$

which implies that

$$|\tilde{Y}(\theta_2^*) - \tilde{Y}(\theta_1^*)| \geq c,$$

and the proof of the claim is finished.

In the same way, if  $\theta_1^* < \theta_2^* < \dots < \theta_{j-1}^* < \theta_j^* < T$ , then

$$|\tilde{Y}(\theta_j^*) - \tilde{Y}(\theta_{j-1}^*)| \geq c.$$

On the other hand, as  $\tilde{Y}$  satisfies (1.1), it is easy to check that

$$E \left[ \sum_{j=1}^{\infty} |\tilde{Y}(\theta_j^*) - \tilde{Y}(\theta_{j-1}^*)|^2 \right] < \infty.$$

As a consequence, there exists  $N(\omega)$  such that  $\theta_N^* = T$ .

Finally, from the choice of  $a^*(\cdot)$ ,

$$\tilde{K}^{a^*(\cdot)} + \tilde{A}^{a^*(\cdot)} = 0.$$

We conclude from (3.12) that

$$\tilde{Y}^{a^*(\cdot)}(0) = U^{a^*(\cdot)}(0),$$

which implies that

$$\tilde{Y}_i(0) = U^{a^*(\cdot)}(0).$$

(iii) The representation for  $\tilde{Y}$  is a combination of both assertions (i) and (ii). This gives the uniqueness of the first component of the adapted solution, and the uniqueness of the other two components of the adapted solution follows then.  $\square$

## 4 Optimal switching of functional SDEs

In this section, we study the optimal switching problem. In order to ensure the existence of optimal switching strategy, we use the weak formulation of the problem. Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and let  $\{\mathcal{F}_t, t \geq 0\}$  be a filtration satisfying the usual conditions. The process  $W$  is an  $\{\mathcal{F}_t, t \geq 0\}$ -Brownian motion on  $R^d$  defined on  $(\Omega, \mathcal{F}, P)$ .

Consider the switched equation

$$X^{a(\cdot)}(t) = x_0 + \int_0^t \sigma(s, X^{a(\cdot)})[dW(s) + b(s, X^{a(\cdot)}, a(s))ds], \quad t \in [0, T] \quad (4.1)$$

and the cost functional

$$J(a(\cdot)) = E \left[ \int_0^T l(s, X^{a(\cdot)}, a(s))ds \right] + E \left[ \sum_{i=1}^{N-1} k(\alpha_{i-1}, \alpha_i) \right]. \quad (4.2)$$

The switching problem is to minimize the cost  $J(a(\cdot))$  with respect to  $a(\cdot)$ , subject to the state equation (4.1).

In the above,  $x_0$  is a fixed point in  $R^d$ .  $\sigma$ ,  $b$  and  $l$  are defined on  $[0, T] \times C([0, T]; R^d)$ ,  $[0, T] \times C([0, T]; R^d) \times \Lambda$  and  $[0, T] \times C([0, T]; R^d) \times \Lambda$ , respectively, with values in  $R^{d \times d}$ ,  $R^d$  and  $R$ , respectively. As in Section 3,  $a(\cdot)$  is an admissible  $(\mathcal{F}_t)_{t \geq 0}$ -adapted switching strategy,  $\mathcal{A}$  is the set of all the admissible  $\{\mathcal{F}_t, t \geq 0\}$ -adapted switching strategies and  $\mathcal{A}^i$  is the subset of  $\mathcal{A}$  consisting of the admissible  $\{\mathcal{F}_t, t \geq 0\}$ -adapted switching strategies starting from the mode  $i$ . We assume that  $\sigma$ ,  $b(\cdot, i)$  and  $l(\cdot, i)$  are progressively measurable functionals on  $C([0, T]; R^d)$  in the following sense:

**Definition 4.1.** Let  $C([0, T]; R^d)$  be the space of continuous functions  $x : [0, T] \rightarrow R^d$ . For  $0 \leq t \leq T$ , define  $\mathcal{G}_t := \sigma(x(s) : 0 \leq s \leq t)$ , and set  $\mathcal{G} := \mathcal{G}_T$ . A progressively measurable functional on  $C([0, T]; R^d)$  is a mapping  $\mu : [0, T] \times C([0, T]; R^d) \rightarrow H$  ( $H$  is some Euclidean space) such that for each fixed  $t \in [0, T]$ ,  $\mu$  restricted to  $[0, t] \times C([0, T]; R^d)$  is  $\mathcal{B}([0, t]) \otimes \mathcal{G}_t / \mathcal{B}(H)$ -measurable.

We assume that  $k$  satisfies Hypothesis 3.1. And we assume also that  $\sigma$ ,  $b$  and  $l$  satisfy the following hypothesis.

**Hypothesis 4.1.** (i)  $\sigma$ ,  $b(\cdot, \cdot, i)$  and  $l(\cdot, \cdot, i)$ ,  $i \in \Lambda$ , are progressively measurable functionals on  $C([0, T]; R^d)$ .

(ii) There exists a constant  $\beta > 0$  such that  $\forall (t, x, x', i) \in [0, T] \times R^d \times R^d \times \Lambda$ ,

$$|b(t, x, i) - b(t, x', i)| + |\sigma(t, x) - \sigma(t, x')| + |l(t, x, i) - l(t, x', i)| \leq \beta \|x - x'\|_{C([0, t]; R^d)}.$$

(iii)  $\sigma$  has a bounded inverse.

(iv)  $b$  is bounded.

Let  $(Y, Z, K)$  be the unique solution in  $(S^2, M^2, N^2)$  of the following RBSDE:

$$\begin{cases} Y_i(t) = \int_t^T \psi(s, X, Z_i(s), i) ds - \int_t^T dK_i(s) - \int_t^T Z_i(s) dW(s), \\ Y_i(s) \leq \min_{j \neq i} \{Y_j(s) + k(i, j)\}, \\ \int_0^T \left( Y_i(s) - \min_{j \neq i} \{Y_j(s) + k(i, j)\} \right) dK_i(s) = 0, \quad i \in \Lambda, \end{cases} \quad (4.3)$$

where  $\psi$  is defined as follows:  $\forall(t, x, z, i) \in [0, T] \times C([0, T]; R^d) \times R^d \times \Lambda$ ,

$$\psi(t, x, z, i) := l(t, x, i) + \langle z, b(t, x, i) \rangle,$$

and  $X$  is the solution to the following functional SDE:

$$X(t) = x_0 + \int_0^t \sigma(s, X) dW(s), \quad t \in [0, T]. \quad (4.4)$$

**Theorem 4.1.** *Let the Hypotheses 3.1 and 4.1 hold. Then*

(i) *For any  $a(\cdot) \in \mathcal{A}^i$ , we have:*

$$J(a(\cdot)) \geq Y_i(0). \quad (4.5)$$

(ii) *There exists an optimal switching strategy  $a^*$ , and a weak solution  $(P^*, W^*, X^*)$ , such that*

$$X^*(t) = x_0 + \int_0^t \sigma(s, X^*) [dW^*(s) + b(s, X^*, a^*(s, X^*))], \quad t \in [0, T], \quad (4.6)$$

and

$$J(a^*(\cdot)) = Y_i(0).$$

**Proof.** (i) For any  $a(\cdot) \in \mathcal{A}^i$ , we set

$$d\bar{P} := \exp \left\{ - \int_0^T b(s, X^{a(\cdot)}, a(s)) dW(s) - \frac{1}{2} \int_0^T |b(s, X^{a(\cdot)}, a(s))|^2 ds \right\} dP.$$

Then  $\bar{W}(t) = W(t) + \int_0^t b(s, X^{a(\cdot)}, a(s)) ds$  is a Brownian motion under the new probability measure  $\bar{P}$ . Let  $(\bar{Y}, \bar{Z}, \bar{K})$  be the solution of the following RBSDE:

$$\begin{cases} \bar{Y}_i(t) &= \int_t^T \psi(s, X^{a(\cdot)}, \bar{Z}_i(s), i) ds - \int_t^T d\bar{K}_i(s) - \int_t^T \bar{Z}_i(s) d\bar{W}(s), \\ \bar{Y}_i(s) &\leq \min_{j \neq i} \{ \bar{Y}_j(s) + k(i, j) \}, \\ &\int_0^T \left( \bar{Y}_i(s) - \min_{j \neq i} \{ \bar{Y}_j(s) + k(i, j) \} \right) d\bar{K}_i(s) = 0, \quad i \in \Lambda. \end{cases} \quad (4.7)$$

Note that since  $X^{a(\cdot)}$  solves (4.1), we have

$$X^{a(\cdot)}(t) = x_0 + \int_0^t \sigma(s, X^{a(\cdot)}) d\bar{W}(s), \quad t \in [0, T].$$

By a classical argument of Yamada-Watanabe, for RBSDE (4.3), the pathwise uniqueness implies the uniqueness in the sense of probability law (see, e.g. [5], for a proof in the framework of BSDE). Hence, we have

$$Y_i(0) = \bar{Y}_i(0), \quad i \in \Lambda. \quad (4.8)$$

Recalling the cost process  $A^{a(\cdot)}$  defined by (3.2), and defining  $\bar{Y}^{a(\cdot)}$ ,  $\bar{Z}^{a(\cdot)}$ ,  $\bar{K}^{a(\cdot)}$  and  $\bar{A}^{a(\cdot)}$  in the same way as (3.8), (3.9), (3.10) and (3.11), we deduce in the same manner as in Section 3 that  $(\bar{Y}^{a(\cdot)}, \bar{Z}^{a(\cdot)})$  is the unique solution of the following BSDE:

$$\begin{aligned} & \bar{Y}^{a(\cdot)}(t) \\ = & A^{a(\cdot)}(T) - A^{a(\cdot)}(t) - [(\bar{K}^{a(\cdot)}(T) + \bar{A}^{a(\cdot)}(T)) - (\bar{K}^{a(\cdot)}(t) + \bar{A}^{a(\cdot)}(t))] \\ & + \int_t^T \psi(s, X^{a(\cdot)}, \bar{Z}^{a(\cdot)}(s), a(s))ds - \int_t^T \bar{Z}^{a(\cdot)}(s)d\bar{W}(s), \quad t \in [0, T]. \end{aligned} \quad (4.9)$$

Since, both  $\bar{K}$  and  $\bar{A}$  are increasing, we deduce that

$$\begin{aligned} & \bar{Y}^{a(\cdot)}(0) \\ \leq & A^{a(\cdot)}(T) + \int_0^T \psi(s, X^{a(\cdot)}, \bar{Z}^{a(\cdot)}(s), a(s))ds - \int_0^T \bar{Z}^{a(\cdot)}(s)[dW(s) + b(s, X^{a(\cdot)}, a(s))] \\ = & A^{a(\cdot)}(T) + \int_0^T l(s, X^{a(\cdot)}, a(s))ds - \int_0^T \bar{Z}^{a(\cdot)}(s)dW(s). \end{aligned} \quad (4.10)$$

From the definition of  $\bar{Y}^{a(\cdot)}$ ,

$$\bar{Y}_i(0) = \bar{Y}^{a(\cdot)}(0). \quad (4.11)$$

From (4.8), (4.11) and (4.10),

$$Y_i(0) \leq A^{a(\cdot)}(T) + \int_0^T l(s, X^{a(\cdot)}, a(s))ds - \int_0^T \bar{Z}^{a(\cdot)}(s)dW(s).$$

Taking expectation with respect to  $P$ , we have

$$Y_i(0) \leq J(a(\cdot)).$$

(ii) Let  $X$  be the solution of SDE (4.4) and  $(Y, Z, K)$  be the solution of RBSDE (4.3). Then,  $Y$  is adapted to the filtration  $\mathcal{F}_t^W = \mathcal{F}_t^X$ , due to (4.4) and Hypothesis 4.1 (iii).

Set  $\theta_0^* = 0$ ,  $\alpha_0^* = i$ . We define the sequence  $\{\theta_j^*, \alpha_j^*\}_{j=1}^\infty$  in an inductive way as follows:

$$\theta_j^* := \inf \{s \geq \theta_{j-1}^* : Y_{\alpha_{j-1}^*}(s) = \min_{l \neq \alpha_{j-1}^*} \{Y_l(s) + k(\alpha_{j-1}^*, l)\} \wedge T, \quad (4.12)$$

and  $\alpha_j^*$  is the  $\mathcal{F}_{\theta_j^*}$ -measurable random variable such that

$$Y_{\alpha_{j-1}^*}(\theta_j^*) = Y_{\alpha_j^*}(\theta_j^*) + k(\alpha_{j-1}^*, \alpha_j^*),$$

with  $j = 1, 2, \dots$ . Then,  $P$ -a.s.  $\omega$ , there exists an integer  $N(\omega)$  such that  $\theta_N^* = T$ . And we define the switching strategy  $a^*$  as follows:

$$a^*(s) := i\chi_{\{0\}}(s) + \sum_{j=1}^N \alpha_{j-1}^* \chi_{(\theta_{j-1}^*, \theta_j^*]}(s), \quad (4.13)$$

is admissible, i.e.,  $a^*(\cdot) \in \mathcal{A}^i$ .  $a^*$  is adapted to the filtration  $\mathcal{F}_t^W = \mathcal{F}_t^X$ , i.e.,  $a^*(t) = a^*(t, X)$ . Setting

$$dP^* := \exp \left( \int_0^t b(s, X, a^*(s, X)) dW(s) - \frac{1}{2} \int_0^t |b(s, X, a^*(s, X))|^2 ds \right) dP,$$

then

$$W^*(t) := W(t) - \int_0^t b(s, X, a^*(s, X)) dW(s), \quad t \in [0, T],$$

is a Brownian motion under the probability measure  $P^*$ , and  $(P^*, W^*, X)$  is a weak solution of (4.1).

Computing  $Y^{a^*(\cdot)}$  as in Section 3, we deduce that

$$\begin{aligned} & Y^{a^*(\cdot)}(0) \\ &= A^{a^*(\cdot)}(T) - (K^{a^*(\cdot)}(T) + \tilde{A}^{a^*(\cdot)}(T)) \\ & \quad + \int_0^T \psi(s, X, Z^{a^*(\cdot)}(s), a^*(s)) ds - \int_0^T Z^{a^*(\cdot)}(s) dW(s), \end{aligned} \quad (4.14)$$

where

$$\begin{aligned} A^{a^*(\cdot)}(s) &= \sum_{j=1}^{N-1} k(\alpha_{j-1}^*, \alpha_j^*) \chi_{[\theta_j^*, T]}(s), \\ Y^{a^*(\cdot)}(s) &= \sum_{i=1}^N Y_{\alpha_{i-1}^*}(s) \chi_{[\theta_{i-1}^*, \theta_i^*)}(s) + Y_{a^*(T)}(s) \chi_{\{T\}}(s), \\ Z^{a^*(\cdot)}(s) &= \sum_{i=1}^N Z_{\alpha_{i-1}^*}(s) \chi_{[\theta_{i-1}^*, \theta_i^*)}(s), \\ K^{a^*(\cdot)}(s) &= \sum_{i=1}^N \int_{\theta_{i-1}^* \wedge s}^{\theta_i^* \wedge s} dK_{\alpha_{i-1}^*}(r), \\ \tilde{A}^{a^*(\cdot)}(s) &= \sum_{i=1}^{N-1} [Y_{\alpha_i}(\theta_i^*) + k(\alpha_{i-1}^*, \alpha_i^*) - Y_{\alpha_{i-1}^*}(\theta_i^*)] \chi_{[\theta_i^*, T]}(s). \end{aligned}$$

From the definition of  $a^*(\cdot)$ ,  $K^{a^*(\cdot)} = 0$  and  $A^{a^*(\cdot)} = 0$ . Hence, from (4.14), it follows that

$$\begin{aligned} & Y^{a^*(\cdot)}(0) \\ &= A^{a^*(\cdot)}(T) + \int_0^T l(s, X, a^*(s)) ds - \int_0^T Z^{a^*(\cdot)}(s) dW^*(s). \end{aligned} \quad (4.15)$$

Taking the expectation with respect to  $P^*$ , we conclude the proof.  $\square$

## 5 System of variational inequalities

In this section, we will show that the RBSDE studied in Sections 2 and 3 allows us to give a probabilistic representation of the solution to a system of variational inequalities. For this purpose, we will put RBSDE (1.1) in a Markovian framework.

Let  $b : [0, T] \times R^d \rightarrow R^d$  and  $\sigma : [0, T] \times R^d \rightarrow R^{d \times d}$  be continuous mappings. We assume:

**Hypothesis 5.1.** *There exists a constant  $C > 0$ , such that for all  $t \in [0, T]$ , and  $(x, x') \in R^d \times R^d$ ,*

$$|b(t, 0)| + |\sigma(t, 0)| \leq C,$$

and

$$|b(t, x) - b(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq C|x - x'|.$$

For each  $(t, x) \in [0, T] \times R^d$ , let  $\{X^{t,x}(s); t \leq s \leq T\}$  be the unique  $R^d$ -valued solution of the SDE:

$$X^{t,x}(s) = x + \int_t^s b(r, X^{t,x}(r))dr + \int_t^s \sigma(r, X^{t,x}(r))dW(r), \quad s \in [t, T].$$

We suppose now that the data  $\xi$  and  $\psi$  of RBSDE (1.1) take the following form:

$$\begin{aligned} \xi_i &= g(X^{t,x}(T), i), \\ \psi(s, y, z, i) &= \psi(s, X^{t,x}(s), y, z, i), \end{aligned}$$

where  $g$  and  $\psi$  are given as follows.

**Hypothesis 5.2.** (i) *For each  $i \in \Lambda$ , the function  $g(\cdot, i) \in C(R^d)$  and has at most polynomial growth at infinity.*

(ii) *For each  $i \in \Lambda$ ,*

$$\psi(\cdot, i) : [0, T] \times R^d \times R \times R^d \rightarrow R$$

*is jointly continuous and there exist two constants  $C > 0$  and  $p \geq 0$  such that*

$$|\psi(t, x, 0, 0, i)| \leq C(1 + |x|^p),$$

$$|\psi(t, x, y, z, i) - \psi(t, x, y', z', i)| \leq C(|y - y'| + |z - z'|),$$

*for  $t \in [0, T]$ ,  $x, z, z' \in R^d$ ,  $y, y' \in R$ , and  $i \in \Lambda$ .*

(iii)  $\forall x \in R^d$ ,

$$g(x) := (g(x, 1), \dots, g(x, m))^T \in \bar{Q}.$$

For each  $t \geq 0$ , we denote by  $\{\mathcal{F}_s^t, t \leq s \leq T\}$  the natural filtration of the Brownian motion  $\{W_s - W_t, t \leq s \leq T\}$ , augmented by the  $P$ -null sets of  $\mathcal{F}$ .

It follows from the results of Sections 2 and 3 that for each  $(t, x)$ , there exists a unique triple  $(Y^{t,x}, Z^{t,x}, K^{t,x})$  in  $S^2 \times M^2 \times N^2$  of  $\{\mathcal{F}_s^t, t \leq s \leq T\}$  progressively measurable processes, which solves the following RBSDE:

$$\left\{ \begin{aligned} Y_i(s) &= g(X^{t,x}(T), i) + \int_s^T \psi(r, X^{t,x}(r), Y_i(r), Z_i(r), i)dr \\ &\quad - \int_s^T dK_i(r) - \int_s^T Z_i(r) dW(r), \quad s \in [0, T]; \\ Y_i(s) &\leq \min_{j \neq i} \{Y_j(s) + k(i, j)\}, \quad s \in [0, T]; \\ &\quad \int_0^T \left( Y_i(s) - \min_{j \neq i} \{Y_j(s) + k(i, j)\} \right) dK_i(s) = 0; \quad i \in \Lambda. \end{aligned} \right. \quad (5.1)$$

We now consider the related system of variational inequalities. Roughly speaking, a solution of the system of variational inequalities is a function  $u : [0, T] \times R^d \rightarrow R^m$  which satisfies:

$$\max \left\{ \begin{aligned} & -\partial_t u_i(t, x) - \mathcal{L}u_i(t, x) - \psi(t, x, u_i(t, x), \nabla u_i(t, x)\sigma(t, x), i), \\ & u_i(t, x) - \min_{j \neq i} (u_j(t, x) + k(i, j)) \end{aligned} \right\} = 0, \quad (5.2)$$

$(t, x, i) \in [0, T] \times R^d \times \Lambda$ , with the terminal condition

$$u_i(T, x) = g(x, i), \quad (x, i) \in R^d \times \Lambda. \quad (5.3)$$

Here, the second-order partial differential operator  $\mathcal{L}$  is given by

$$\mathcal{L} := \frac{1}{2} \sum_{j,l=1}^d (\sigma \sigma^T(t, x))_{j,l} \frac{\partial^2}{\partial x_j \partial x_l} + \sum_{j=1}^d b_j(t, x) \frac{\partial}{\partial x_j}.$$

More precisely, we shall consider solution of (5.2) in the viscosity sense. It will be convenient for the sequel to define the notion of viscosity solution in the language of sub- and super-jets (see, e.g., [4]). Below,  $S(d)$  will denote the set of  $d \times d$  symmetric nonnegative matrices.

**Definition 5.1.** Let  $u \in C((0, T) \times R^d; R)$  and  $(t, x) \in (0, T) \times R^d$ . We denote by  $\mathcal{P}^{2,+}u(t, x)$  [the “parabolic superjet” of  $u$  at  $(t, x)$ ] the set of triples  $(p, q, X) \in R \times R^d \times S(d)$  which are such that

$$u(s, y) \leq u(t, x) + p(s - t) + \langle q, y - x \rangle + \frac{1}{2} \langle X(y - x), y - x \rangle + o(|s - t| + |y - x|^2).$$

Similarly, we denote by  $\mathcal{P}^{2,-}u(t, x)$  [the “parabolic subjet” of  $u$  at  $(t, x)$ ] the set of triples  $(p, q, X) \in R \times R^d \times S(d)$  which are such that

$$u(s, y) \geq u(t, x) + p(s - t) + \langle q, y - x \rangle + \frac{1}{2} \langle X(y - x), y - x \rangle + o(|s - t| + |y - x|^2).$$

We can now give the definition of a viscosity solution of the system of variational inequalities (5.2) and (5.3).

**Definition 5.2.**  $u \in C([0, T] \times R^d; R^m)$  is called a viscosity subsolution (resp., supersolution) of (5.2) and (5.3) if  $u_i(T, x) \leq$  (resp.  $\geq$ )  $g(x, i)$ ,  $(x, i) \in R^d \times \Lambda$ , and at any point  $(t, x, i) \in (0, T) \times R^d \times \Lambda$ , for any  $(p, q, X) \in \mathcal{P}^{2,+}u_i(t, x)$  (resp.,  $\mathcal{P}^{2,-}u_i(t, x)$ ),

$$\max \left\{ \begin{aligned} & -p - \frac{1}{2} \text{Tr}(\sigma \sigma^T(t, x)X) - \langle b(t, x), q \rangle - \psi(t, x, u_i(t, x), q\sigma(t, x), i), \\ & u_i(t, x) - \min_{j \neq i} (u_j(t, x) + k(i, j)) \end{aligned} \right\} \leq \text{ (resp., } \geq \text{) } 0.$$

$u \in C([0, T] \times R^d; R^m)$  is called a viscosity solution of (5.2) and (5.3) if it is both a subsolution and a supersolution of (5.2) and (5.3).



We now define

$$u_i(t, x) := Y_i^{t,x}(t), \quad (t, x, i) \in [0, T] \times R^d \times \Lambda; \quad u := (u_1, u_2, \dots, u_m)^T. \quad (5.4)$$

Note that  $u$  is deterministic.

**Lemma 5.1.** *Let the Hypotheses 3.1, 5.1 and 5.2 hold. For each  $i \in \Lambda$ ,  $u_i \in C([0, T] \times R^d; R)$ .*

**Proof:** For  $(t, x) \in [0, T] \times R^d$ ,  $i \in \Lambda$  and  $a(\cdot) \in \mathcal{A}_t^i$ , let  $(U_{t,x}^{a(\cdot)}, V_{t,x}^{a(\cdot)})$  be the unique solution of the following switched BSDE:

$$\begin{aligned} U(s) &= g(X^{t,x}(T), a(T)) + A^{a(\cdot)}(T) - A^{a(\cdot)}(s) \\ &\quad + \int_s^T \psi(r, X^{t,x}(r), U(r), V(r), a(r)) dr \\ &\quad - \int_s^T V(r) dW(r), \quad s \in [0, T]. \end{aligned} \quad (5.5)$$

From Theorem 3.1, we have

$$u_i(t, x) = \inf_{a(\cdot) \in \mathcal{A}_t^i} U_{t,x}^{a(\cdot)}(t), \quad (t, x, i) \in [0, T] \times R^d \times \Lambda.$$

By some classical stability arguments, we obtain the continuity of  $u_i$ .  $\square$

**Theorem 5.1.** *Let the Hypotheses 3.1, 5.1 and 5.2 be true. The function  $u$  given by (5.4) is the viscosity solution of the system of variational inequalities (5.2) and (5.3).*

**Proof:** We are going to approximate RBSDE (5.1) by penalization, which was studied in Section 2. For each  $(t, x) \in [0, T] \times R^d$ , let  $\{^n Y^{t,x}(s), ^n Z^{t,x}(s), t \leq s \leq T\}$  denote the solution of the penalized BSDE:

$$Y_i(s) = g(X^{t,x}(T), i) + \int_s^T \psi^n(r, X^{t,x}(r), Y(r), Z_i(r), i) dr - \int_s^T Z_i(r) dW(r), \quad (5.6)$$

where for  $(t, x, y, z, i) \in [0, T] \times R^d \times R^m \times R^d \times \Lambda$ ,

$$\psi^n(t, x, y, z, i) := \psi(t, x, y_i, z, i) - n \sum_{j \neq i} (y_i - y_j - k(i, j))^+.$$

It is known from [15] and [1] that

$$u^n(t, x) := ^n Y^{t,x}(t), \quad (t, x) \in [0, T] \times R^d,$$

is the viscosity solution to the following system of parabolic PDEs:

$$\begin{aligned} -\partial_t u_i^n(t, x) - \mathcal{L}u_i^n(t, x) &= \psi^n(t, x, u^n(t, x), \nabla u_i^n(t, x) \sigma(t, x), i) = 0, \\ u_i^n(T, x) &= g(x, i), \quad (t, x, i) \in [0, T] \times R^d \times \Lambda. \end{aligned} \quad (5.7)$$

However, from the results of Section 2, for each  $(t, x, i) \in [0, T] \times R^d \times \Lambda$ ,

$$u_i^n(t, x) \downarrow u_i(t, x), \text{ as } n \rightarrow \infty.$$

Since  $u^n$  and  $u$  are continuous, it follows from Dini's theorem that the above convergence is uniform on compacts.

We now show that  $u$  is a subsolution of (5.2) and (5.3). Let  $(t, x, i)$  be a point in  $[0, T] \times R^d \times \Lambda$ . Since  $u$  is defined by (5.4),

$$u_i(t, x) \leq \min_{j \neq i} (u_j(t, x) + k(i, j)),$$

and

$$u_i(T, x) = g(x, i).$$

Let  $(t, x, i) \in (0, T) \times R^d \times \Lambda$  and  $(p, q, X) \in \mathcal{P}^{2,+} u_i(t, x)$ . From Lemma 6.1 in [4], there exist sequences

$$\begin{aligned} n_l &\rightarrow +\infty, \\ (t_l, x_l) &\rightarrow (t, x), \\ (p_l, q_l, X_l) &\in \mathcal{P}^{2,+} u_i^{n_l}(t, x), \end{aligned}$$

such that

$$(p_l, q_l, X_l) \rightarrow (p, q, X).$$

On the other hand, for any  $l$ , from (5.7),

$$-p_l - \frac{1}{2} \text{Tr}(\sigma \sigma^T(t, x) X_l) - \langle b(t, x), q_l \rangle - \psi(t, x, u_i^{n_l}(t, x), q_l \sigma(t, x), i) \leq 0.$$

Hence, taking the limit as  $j \rightarrow \infty$  in the last inequality yields:

$$-p - \frac{1}{2} \text{Tr}(\sigma \sigma^T(t, x) X) - \langle b(t, x), q \rangle - \psi(t, x, u_i(t, x), q \sigma(t, x), i) \leq 0.$$

We have proved that  $u$  is a subsolution of (5.2).

We conclude by showing that  $u$  is a supersolution of (5.2) and (5.3). Let  $(t, x, i) \in (0, T) \times R^d \times \Lambda$  be a point at which  $u_i(t, x) < \min_{j \neq i} (u_j(t, x) + l(i, j))$ , and let  $(p, q, X) \in \mathcal{P}^{2,-} u_i(t, x)$ . Again from Lemma 6.1 in [4], there exist sequences

$$\begin{aligned} n_l &\rightarrow +\infty, \\ (t_l, x_l) &\rightarrow (t, x), \\ (p_l, q_l, X_l) &\in \mathcal{P}^{2,-} u_i^{n_l}(t, x), \end{aligned}$$

such that

$$(p_l, q_l, X_l) \rightarrow (p, q, X).$$

On the other hand, for any  $l$ , from (5.7),

$$-p_l - \frac{1}{2} \text{Tr}(\sigma \sigma^T(t, x) X_l) - \langle b(t, x), q_l \rangle - \psi^{n_l}(t, x, u_i^{n_l}(t, x), q_l \sigma(t, x), i) \geq 0.$$

From the assumption that  $u_i(t, x) < \min_{j \neq i} (u_j(t, x) + l(i, j))$  and the uniform convergence of  $u^n$ , it follows that for  $j$  large enough,  $u_i^{n_l}(t_l, x_l) < \min_{j \neq i} (u_j^{n_l}(t_l, x_l) + l(i, j))$ ; hence, taking the limit as  $j \rightarrow \infty$  in the last inequality yields:

$$-p - \frac{1}{2} \text{Tr}(\sigma \sigma^T(t, x) X) - \langle b(t, x), q \rangle - \psi(t, x, u_i(t, x), q \sigma(t, x), i) \geq 0.$$

We have proved that  $u$  is a supersolution of (5.2) and (5.3).

Hence,  $u$  is a viscosity solution of (5.2) and (5.3). □

**Remark 5.1.** *Uniqueness of viscosity solution to (5.2) and (5.3) follows from classical arguments. See, e.g. [4] and [18].*

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